W5. Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be positive real sequences such that

$$\lim_{n\to\infty}\frac{a_{n+1}-a_n}{n}=a\in\mathbb{R}_+^* \text{ and } \lim_{n\to\infty}\frac{b_{n+1}}{nb_n}=b\in\mathbb{R}_+^*.$$

Compute

$$\lim_{n\to\infty}\left(\frac{a_{n+1}}{\sqrt[n+1]{b_{n+1}}}-\frac{a_n}{\sqrt[n]{b_n}}\right).$$

D.M Bătineţu-Giurgiu and Neculai Stanciu.

Solution by Arkady Alt, San Jose, California, USA.

First note that by Stolz-Cesaro Theorem $\lim\limits_{n\to\infty}\frac{a_{n+1}-a_n}{n}=a$ implies

$\lim\limits_{n\to\infty}\dfrac{a_n}{\frac{n(n+1)}{2}}=a\iff\lim\limits_{n\to\infty}\dfrac{a_n}{n^2}=\dfrac{a}{2}$.

Also, since $\lim\limits_{n\to\infty}\frac{b_{n+1}}{nb_n}=b$ and $\frac{b_{n+1}}{(n+1)b_n}=\frac{b_{n+1}}{nb_n}\cdot\frac{n}{n+1}$ then $\lim\limits_{n\to\infty}\frac{b_{n+1}}{(n+1)b_n}=b$ that is $\lim\limits_{n\to\infty}\frac{b_{n+1}/((n+1)!)}{b_n/n!}=b$ and, therefore, $\lim\limits_{n\to\infty}\sqrt[n]{\frac{b_n}{n!}}=b$ by Multiplicative Stolz-Cesaro Theorem. Taking in account that $\lim\limits_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\frac{1}{e}$ we obtain

$\lim\limits_{n\to\infty}\dfrac{\sqrt[n]{b_n}}{n}=\lim\limits_{n\to\infty}\left(\sqrt[n]{\dfrac{b_n}{n!}}\cdot\dfrac{\sqrt[n]{n!}}{n}\right)=\dfrac{b}{e}$.

Thus, $\lim\limits_{n\to\infty}\left(\dfrac{a_{n+1}}{\sqrt[n+1]{b_{n+1}}}-\dfrac{a_n}{\sqrt[n]{b_n}}\right)=\lim\limits_{n\to\infty}\dfrac{a_n}{\sqrt[n]{b_n}}\left(\dfrac{a_{n+1}\sqrt[n]{b_n}}{a_n\sqrt[n+1]{b_{n+1}}}-1\right)=$

$\lim\limits_{n\to\infty}\left(\dfrac{a_n}{n^2}\cdot\dfrac{n}{\sqrt[n]{b_n}}\cdot n\left(\dfrac{a_{n+1}\sqrt[n]{b_n}}{a_n\sqrt[n+1]{b_{n+1}}}-1\right)\right)=\lim\limits_{n\to\infty}\dfrac{a_n}{n^2}\cdot\lim\limits_{n\to\infty}\dfrac{n}{\sqrt[n]{b_n}}\cdot\lim\limits_{n\to\infty}n\left(\dfrac{a_{n+1}\sqrt[n]{b_n}}{a_n\sqrt[n+1]{b_{n+1}}}-1\right)=$

$\dfrac{ab}{2e}\lim\limits_{n\to\infty}n\left(\dfrac{a_{n+1}\sqrt[n]{b_n}}{a_n\sqrt[n+1]{b_{n+1}}}-1\right)$. Let $\alpha_n:=\dfrac{a_{n+1}\sqrt[n]{b_n}}{a_n\sqrt[n+1]{b_{n+1}}}-1$. Since $\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n}=$

$\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{(n+1)^2}\cdot\dfrac{n^2}{a_n}\cdot\dfrac{(n+1)^2}{n^2}=b\cdot\dfrac{1}{b}\cdot 1=1$ and $\lim\limits_{n\to\infty}\dfrac{\sqrt[n]{b_n}}{\sqrt[n+1]{b_{n+1}}}=$

$\lim\limits_{n\to\infty}\left(\dfrac{\sqrt[n]{b_n}}{n}\cdot\dfrac{n+1}{\sqrt[n+1]{b_{n+1}}}\cdot\dfrac{n}{n+1}\right)=\dfrac{b}{e}\cdot\dfrac{e}{b}\cdot 1=1$ then $\lim\limits_{n\to\infty}\alpha_n=0$ and, therefore,

$\lim\limits_{n\to\infty}n\left(\dfrac{a_{n+1}\sqrt[n]{b_n}}{a_n\sqrt[n+1]{b_{n+1}}}-1\right)=\lim\limits_{n\to\infty}\left(n\ln\left(\dfrac{a_{n+1}\sqrt[n]{b_n}}{a_n\sqrt[n+1]{b_{n+1}}}\right)\cdot\dfrac{\alpha_n}{\ln(1+\alpha_n)}\right)=$

$\lim\limits_{n\to\infty}\left(n\ln\left(\dfrac{a_{n+1}\sqrt[n]{b_n}}{a_n\sqrt[n+1]{b_{n+1}}}\right)\right)\cdot\lim\limits_{n\to\infty}\dfrac{\alpha_n}{\ln(1+\alpha_n)}=\ln\left(\lim\limits_{n\to\infty}\left(\dfrac{a_{n+1}\sqrt[n]{b_n}}{a_n\sqrt[n+1]{b_{n+1}}}\right)^n\right)$.

$\lim\limits_{n\to\infty}\left(\dfrac{a_{n+1}\sqrt[n]{b_n}}{a_n\sqrt[n+1]{b_{n+1}}}\right)=\lim\limits_{n\to\infty}\left(\left(\dfrac{a_{n+1}}{a_n}\right)^n\cdot\dfrac{nb_n}{b_{n+1}}\cdot\dfrac{\sqrt[n+1]{b_{n+1}}}{n+1}\cdot\dfrac{n+1}{n}\right)=$

$\lim\limits_{n\to\infty}\left(\dfrac{a_{n+1}}{a_n}\right)^n\cdot\lim\limits_{n\to\infty}\dfrac{nb_n}{b_{n+1}}\cdot\lim\limits_{n\to\infty}\dfrac{\sqrt[n+1]{b_{n+1}}}{n+1}\cdot\lim\limits_{n\to\infty}\dfrac{n+1}{n}=\lim\limits_{n\to\infty}\left(\dfrac{a_{n+1}}{a_n}\right)^n\cdot\dfrac{1}{b}\cdot\dfrac{b}{e}=$

$\dfrac{1}{e}\lim\limits_{n\to\infty}\left(\dfrac{a_{n+1}}{a_n}\right)^n$. Thus, $\lim\limits_{n\to\infty}\left(\dfrac{a_{n+1}}{\sqrt[n+1]{b_{n+1}}}-\dfrac{a_n}{\sqrt[n]{b_n}}\right)=\dfrac{ab}{2e}\lim\limits_{n\to\infty}n\left(\dfrac{a_{n+1}\sqrt[n]{b_n}}{a_n\sqrt[n+1]{b_{n+1}}}-1\right)=$

$\dfrac{ab}{2e^2}\lim\limits_{n\to\infty}\left(\dfrac{a_{n+1}}{a_n}\right)^n$ and remains calculate $\lim\limits_{n\to\infty}\left(\dfrac{a_{n+1}}{a_n}\right)^n$.

Let $\beta_n := \dfrac{a_{n+1}}{a_n} - 1$. Since $\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n} = 1$ then $\lim\limits_{n\to\infty}\beta_n = 0$, $\lim\limits_{n\to\infty}\left(\dfrac{a_{n+1}}{a_n}\right)^n =$
$\lim\limits_{n\to\infty}\left((1+\beta_n)^{\frac{1}{\beta_n}}\right)^{n\beta_n}$ and we have $\lim\limits_{n\to\infty} n\beta_n = \lim\limits_{n\to\infty} n\left(\dfrac{a_{n+1}}{a_n} - 1\right) =$
$\lim\limits_{n\to\infty}\left(\dfrac{n^2}{a_n}\cdot\dfrac{a_{n+1}-a_n}{n}\right) = \lim\limits_{n\to\infty}\dfrac{n^2}{a_n}\cdot\lim\limits_{n\to\infty}\dfrac{a_{n+1}-a_n}{n} = \dfrac{2}{a}\cdot a = 2.$

Hence, $\lim\limits_{n\to\infty}\left(\dfrac{a_{n+1}}{a_n}\right)^n = e^2$ and, therefore, $\lim\limits_{n\to\infty}\left(\dfrac{a_{n+1}}{\sqrt[n+1]{b_{n+1}}} - \dfrac{a_n}{\sqrt[n]{b_n}}\right) = \dfrac{ab}{2e^2}\cdot e^2 = \dfrac{ab}{2}.$